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Uniform asymptotic solutions for potential flow about a slender body of revolution

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The general problem of potential flow past a slender body of revolution is considered. The flow incident on the body is described by an arbitrary potential function and hence the results presented here extend those obtained by Handelsman & Keller (1967*a*). The part of the potential due to the presence of the body is represented as a superposition of potentials due to point singularities (sources, dipoles and higher-order singularities) distributed along a segment of the axis of the body inside the body. The boundary condition on the body leads to a linear integral equation for the density of the singularities. The complete uniform asymptotic expansion of the solution of this equation, as well as the extent of the distribution, is obtained using the method of Handelsman & Keller. The special case of transverse incident flow is considered in detail. Complete expansions for the dipole moment of the distribution and the virtual mass of the body are obtained. Some general comments on the method of Handelsman & Keller are given, and may be useful to others wishing to use their method.

1. Introduction

We wish to consider the general case of potential flow past a slender body of revolution. The results presented here extend those obtained in Handelsman & Keller (1967*a*), where only axially symmetric flows past the body were considered. They represented the disturbance potential ϕ^b due to the presence of the body as a superposition of potentials due to point sources distributed along a segment of the axis of the body inside the body. They then presented a special method for obtaining the uniform asymptotic solution of an integral equation resulting from the boundary condition for the problem. Their method (see also Fraenkel 1969) has been used by others, e.g. Tillett (1970), to describe Stokes flow past a slender body of revolution and also by Geer & Keller (1968) and Geer (1974) to solve some two-dimensional flow problems.

To illustrate the method used here, we first consider in detail the case when the incident flow, which in general is described by a potential ϕ^0, is a uniform flow transverse to the axis of the body. For this case, the part of the potential due to the presence of the body is represented as a superposition of potentials due to point dipoles oriented in the direction of the flow and distributed along a segment of the axis of the body inside the body. The boundary condition on the body then leads to a linear integral equation for the dipole strength density. Using the method of Handelsman & Keller, the complete uniform asymptotic expansion of the solution of this equation, as well as the extent of the distribution, is obtained. Complete expansions for the total dipole moment of the distribution and the virtual mass of the body are then found.

These results are then generalized to the case of a general incident flow described by an arbitrary potential ϕ^0. The corresponding disturbance potential ϕ^b is represented as a superposition of potentials due to appropriate (in general, higher-order) singularities distributed along a segment of the axis of the body inside the body. Again the boundary condition on the body leads to an integral equation, which can be solved using the method of Handelsman & Keller. It is interesting that the extent of the distribution for all of these higher-order singularities is the same as for the source distribution in the axially symmetric case.

Finally, some general comments and observations are made about the general method of Handelsman & Keller, and may be helpful to others who would like to use their method to solve related or other problems.

2. Formulation of the problem

We introduce cylindrical co-ordinates (r, θ, z) in the usual way, with the z axis coinciding with the axis of the body. Let the equation of the surface of the body be $r = \epsilon[S(z)]^{\frac{1}{2}}$, $0 \leqslant z \leqslant 1$, where $\max S(z) = 1$. We shall assume that $S(z)$ is analytic on $0 \leqslant z \leqslant 1$ with $S(0) = 0 = S(1)$ and can be expanded in power series about the end points as follows:

$$S(z) = \sum_{n=1}^{\infty} c_n z^n, \quad c_n = \frac{S^{(n)}(0)}{n\,!}, \tag{2.1}$$

$$S(z) = \sum_{n=1}^{\infty} d_n (1-z)^n, \quad d_n = \frac{(-1)^n S^{(n)}(1)}{n\,!}. \tag{2.2}$$

We shall assume that $c_1 \neq 0 \neq d_1$, i.e. that the radii of curvature at the ends of the body are non-zero.

We now seek a function ϕ^b which is harmonic in the region outside the body such that $\phi^0 + \phi^b$ has vanishing normal derivative on the body. Here ϕ^0 is given and is harmonic in a neighbourhood of the body, while ϕ^b must vanish at infinity. ϕ^0 represents the potential of the oncoming stream and hence, to begin with, we set $\phi^0 = r\cos\theta$.

We now wish to represent ϕ^b as a superposition of potentials due to point dipoles distributed along a segment of the axis inside the body. Thus we set

$$\phi^b(r,\theta,z) = -\frac{1}{4\pi}\int_{\alpha(\epsilon)}^{\beta(\epsilon)} \frac{r\cos\theta}{[r^2+(z-\xi)^2]^{\frac{3}{2}}}\,\mu(\xi,\epsilon)\,d\xi, \tag{2.3}$$

where $\mu(\xi,\epsilon)$ is the unknown dipole strength density. Here $\alpha(\epsilon)$ and $\beta(\epsilon)$, which determine the extent of the dipole distribution, must be found in addition to $\mu(\xi,\epsilon)$. They must satisfy the inequalities $0 < \alpha(\epsilon) < \beta(\epsilon) < 1$. ϕ^b as defined by (2.3) is harmonic outside the body and vanishes at infinity. The condition that the normal derivative of $\phi^0+\phi^b$ vanish on the body, when used with (2.3), becomes

$$4\pi = \int_{\alpha(\epsilon)}^{\beta(\epsilon)} \left\{\frac{1}{[(z-\xi)^2+\epsilon^2 S(z)]^{\frac{3}{2}}} - \tfrac{3}{2}\epsilon^2 \frac{2S(z)-S'(z)\,(z-\xi)}{[(z-\xi)^2+\epsilon^2 S(z)]^{\frac{5}{2}}}\right\} \mu(\xi,\epsilon)\,d\xi. \qquad (2.4)$$

Equation (2.4) is a linear integral equation, from which we shall determine $\mu(\xi,\epsilon)$, as well as $\alpha(\epsilon)$ and $\beta(\epsilon)$.

3. Asymptotic solution of the integral equation

Before we solve (2.4), we look first at an instructive example. If the slender body is the ellipsoid of revolution given by $r = \epsilon[z(1-z)]^{\frac{1}{2}}$, the problem outlined in §2 can be solved exactly (see Lamb 1932, pp. 152–153). Once the solution is known, the dipole strength density can be recovered by a method similar to that used in Geer (1974). The result for the ellipsoid of revolution is

$$\mu(z,\epsilon) = f(\epsilon)\,(\beta(\epsilon)-z)\,(z-\alpha(\epsilon)), \qquad (3.1)$$

where $f(\epsilon)$ is a function of ϵ alone and $\alpha(\epsilon)$ and $\beta(\epsilon)$ are the foci of the ellipsoid. Thus, in particular, we see that μ vanishes at $\alpha(\epsilon)$ and $\beta(\epsilon)$.

Using this example as a guide, we look for a solution for $\mu(z,\epsilon)$ of (2.4) of the form

$$\mu(z,\epsilon) = (\beta(\epsilon)-z)\,(z-\alpha(\epsilon))f(z,\epsilon), \qquad (3.2)$$

where $f(z,\epsilon)$ is to be found. We shall assume that f is a regular function of z. If we substitute (3.2) into (2.4) and then expand the right side of (2.4) asymptotically about $\epsilon = 0$, using the method of Handelsman & Keller, and not taking into account the dependence of f on ϵ, (2.4) becomes

$$4\pi \sim \frac{2z(z-1)}{S(z)} f(z,\epsilon)\,\epsilon^{-2} + \sum_{j=0}^{\infty} \epsilon^{2j}(L_j + \log(\epsilon)\,N_j) f(z,\epsilon). \qquad (3.3)$$

Here the L_j and N_j are linear operators which are defined explicitly in appendix A.

Now (3.3) suggests that we look for an asymptotic expansion for $f(z,\epsilon)$ of the form

$$f(z,\epsilon) \sim \epsilon^2 \sum_{n=0}^{\infty} \sum_{m=0}^{n} \epsilon^{2n} (\log\epsilon)^m f_{n,m}(z). \qquad (3.4)$$

Here the $f_{n,m}$ are functions of z, independent of ϵ, which are to be determined. Substituting (3.4) into (3.3) and then equating the coefficients of like terms of the form $\epsilon^{2n}(\log\epsilon)^m$ on each side of (3.3), we obtain the following results:

$$f_{0,0} = -2\pi S(z)/z(1-z), \qquad (3.5)$$

$$f_{k,0}(z) = \frac{S(z)}{2z(1-z)} \sum_{n=0}^{k-1} L_{k-n-1} f_{n,0}(z), \quad k \geqslant 1, \qquad (3.6)$$

and $$f_{k,m}(z) = \frac{S(z)}{2z(1-z)} \sum_{n=m}^{k-1} L_{k-n-1} f_{n,m}(z) + \sum_{n=m-1}^{k-1} N_{k-n-1} f_{n,m-1}(z), \quad k \geqslant 1, \quad 1 \leqslant m \leqslant k. \qquad (3.7)$$

From (3.5)–(3.7) we see that the $f_{n,m}$ can be determined recursively. Also, since $f_{0,0}$ is a regular function of z on $0 \leqslant z \leqslant 1$, it follows from (3.6) and (3.7) that all of the $f_{n,m}$ will be regular on $0 \leqslant z \leqslant 1$ if each $L_j F(z)$ and $N_j F(z)$ is regular whenever $F(z)$ is regular. Using the explicit expressions for the operators L_j and N_j in appendix A, this last requirement leads to exactly the same requirements on $\alpha(\epsilon)$ and $\beta(\epsilon)$ as those in Handelsman & Keller. Hence, $\alpha(\epsilon)$ and $\beta(\epsilon)$ are the same as the functions found in Handelsman & Keller and their leading terms are given by

$$\alpha(\epsilon)/c_1 = (\tfrac{1}{2}\epsilon)^2 - c_2(\tfrac{1}{2}\epsilon)^4 + (c_1c_3 + 2c_2^2)(\tfrac{1}{2}\epsilon)^6 - (c_1^2c_4 + 7c_1c_2c_3 + 5c_2^3)(\tfrac{1}{2}\epsilon)^8 + O(\epsilon^{10}) \quad (3.8)$$

and
$$(1-\beta(\epsilon))/d_1 = (\tfrac{1}{2}\epsilon)^2 - d_2(\tfrac{1}{2}\epsilon)^4 + (d_1d_3 + 2d_2^2)(\tfrac{1}{2}\epsilon)^6 - (d_1^2d_4 + 7d_1d_2d_3 + 5d_2^3)(\tfrac{1}{2}\epsilon)^8 + O(\epsilon^{10}). \quad (3.9)$$

Using the formulae (A 3) and (A 4) of appendix A, we find from (3.6) and (3.7) that

$$\begin{aligned} f_{1,0}(z) = \frac{S(z)}{2z(1-z)}\Big\{ & \tfrac{1}{4}(1-10z+10z^2)f''_{0,0}(z) + 5f'_{0,0}(z)(2z-1) \\ & + \Big[4 + \Big(\frac{d}{dz}[(1-2z)J(z)]\Big)\Big/2J(z) \\ & + \{[z(1-z)S'(0) - S(z)(1-2z)][S(z)+z(1-z)S'(1)]\}/2S(z)z^2(1-z)^2\Big]f_{0,0}(z) \\ & + \Big(\int_{-z}^{0} - \int_{0}^{1-z}\Big)F_2(z,v)\,dv + \frac{d}{dz}\Big(\int_{0}^{1-z} - \int_{-z}^{0}\Big)F_1(z,v)\,dv\Big\}, \end{aligned} \quad (3.10)$$

where
$$F_m(z,v) = v^{-m-1}\Big\{f_{0,0}(z+v) - \sum_{j=0}^{m}\frac{f_{0,0}^{(j)}(z)\,v^j}{j!}\Big\}(1-z-v)(v+z)$$

and
$$f_{1,1}(z) = \frac{S(z)}{2z(1-z)}\Big\{2f_{0,0}(z) - z(1-z)J(z)f''_{0,0}(z) - 2f'_{0,0}(z)\Big[(1-2z)J(z) + \frac{d}{dz}[z(1-z)J(z)]\Big]\Big\}, \quad (3.11)$$

where $J(z) = \log[4z(1-z)/S(z)]$.

When (3.5), (3.10) and (3.11) are used in (3.4) and then (3.4) is used in (3.2), they yield the asymptotic expansion of $\mu(z,\epsilon)$ up to terms of order $\epsilon^6(\log\epsilon)^2$.

4. Dipole moment and virtual mass

From (2.3) and (3.2), it follows that the total dipole moment D of our distribution of dipoles is given by

$$D = -\frac{1}{4\pi}\int_{\alpha(\epsilon)}^{\beta(\epsilon)} f(\xi,\epsilon)(\beta(\epsilon)-\xi)(\xi-\alpha(\epsilon))\,d\xi. \quad (4.1)$$

Inserting (3.4) into (4.1) and then expanding the resulting integrals in Taylor series about $\epsilon = 0$, we find the following expansion for D:

$$D \sim \frac{-\epsilon^2}{4\pi}\sum_{k=0}^{\infty}\epsilon^{2k}\sum_{m=0}^{k}(\log\epsilon)^m \times \sum_{p=m}^{k}\frac{1}{(k-p)!}\Big[\Big(\frac{d}{d\epsilon^2}\Big)^{k-p}\int_{\alpha(\epsilon)}^{\beta(\epsilon)} f_{p,m}(\xi)(\beta(\epsilon)-\xi)(\xi-\alpha(\epsilon))\,d\xi\Big]_{\epsilon=0}. \quad (4.2)$$

Once the dipole moment D has been found, we can compute the virtual mass M of the body by using the formula of Schiffer & Szego (1949):

$$M = 4\pi\rho D U^{-1} - \rho V. \tag{4.3}$$

Here ρ is the density of the fluid, V is the volume of the body and U is the speed of the incident stream at infinity. Setting $U = 1$ and using (4.2), (4.3) becomes

$$\begin{aligned}\rho^{-1}M = {} & \epsilon^2\pi\int_0^1 S(\xi)\,d\xi + \epsilon^4\log\epsilon\int_0^1 f_{1,1}(\xi)\,\xi(1-\xi)\,d\xi \\ & + \epsilon^4\left\{\int_0^1 f_{1,0}(\xi)\,(1-\xi)\,\xi\,d\xi - \frac{\pi}{2}\int_0^1 \frac{S(\xi)}{\xi(1-\xi)}\,[\xi S'(1) - (1-\xi)\,S'(0)]\,d\xi\right\} \\ & + O(\epsilon^6\,(\log\epsilon)^2).\end{aligned} \tag{4.4}$$

In (4.4), $f_{1,0}$ and $f_{1,1}$ are given by (3.10) and (3.11), respectively.

5. Body in a non-uniform flow field

We can now easily generalize our results of the previous sections to the case of a general oncoming flow. Let the oncoming flow be described by the velocity potential ϕ^0. We assume that ϕ^0 is harmonic in a neighbourhood of the body, so that, for $0 \leqslant z \leqslant 1$ and for small r, ϕ^0 can be expanded in a series of the form

$$\phi^0(r,\theta,z) = \tfrac{1}{2}A_0(r^2,z) + \sum_{n=1}^{\infty}\{r^n A_n(r^2,z)\cos n\theta + r^n B_n(r^2,z)\sin n\theta\}. \tag{5.1}$$

In (5.1), each A_n and B_n is a regular function of r^2 and z in a neighbourhood of $r = 0, 0 \leqslant z \leqslant 1$. By superposition, we need consider only the case when ϕ^0 has the form of one of the terms in the series (5.1) and hence we set

$$\phi^0(r,\theta,z) = r^n\psi(r^2,z)\,e^{in\theta} \quad (n \geqslant 0), \tag{5.2}$$

where ψ is a prescribed function, regular in r^2 and z near $r = 0$ and $0 \leqslant z \leqslant 1$. We represent the corresponding ϕ^b as a superposition of potentials due to appropriate (higher-order) singularities distributed along a segment of the axis inside the body. Thus, we set

$$\phi^b(r,\theta,z) = -\frac{1}{4\pi}\int_{\alpha(\epsilon)}^{\beta(\epsilon)} \frac{r^n e^{in\theta}}{[r^2+(z-\xi)^2]^{n+\frac{1}{2}}}\,(\xi-\alpha(\epsilon))^n\,(\beta(\epsilon)-\xi)^n f(\xi,\epsilon)\,d\xi. \tag{5.3}$$

In (5.2), $\alpha(\epsilon)$, $\beta(\epsilon)$ and $f(\xi,\epsilon)$ play the same role as in §§2 and 3 above. ϕ^b as defined by (5.3) is harmonic outside the body and vanishes at infinity. The requirement that the normal derivative of $\phi^0+\phi^b$ vanish on the body, when used with (5.2) and (5.3), becomes

$$\begin{aligned}& 2n\psi(\epsilon^2S(z),z) + 4\epsilon^2S(z)\frac{\partial\psi}{\partial r^2}(\epsilon^2S(z),z) - \epsilon^2S'(z)\frac{\partial\psi}{\partial z}(\epsilon^2S(z),z) \\ & \quad = \frac{1}{4\pi}\int_{\alpha(\epsilon)}^{\beta(\epsilon)} \frac{2n(z-\xi)^2 - 2(n+1)\,\epsilon^2S(z) + (2n+1)\,\epsilon^2S'(z)\,(z-\xi)}{(\epsilon^2S(z)+(z-\xi)^2)^{n+\frac{3}{2}}} \\ & \qquad \times(\xi-\alpha(\epsilon))^n\,(\beta(\epsilon)-\xi)^n f(\xi,\epsilon)\,d\xi.\end{aligned} \tag{5.4}$$

Equation (5.4) is a linear integral equation from which we can determine $\alpha(\epsilon)$, $\beta(\epsilon)$ and $f(\xi,\epsilon)$. The case $n = 0$ has been treated by Handelsman & Keller (1967*a*) and the case $n = 1$ with $\psi = 1$ has been treated above.

To find $f(\xi,\epsilon)$, we again expand both sides of our integral equation asymptotically about $\epsilon = 0$ without taking into account the dependence of f on ϵ. The left side of (5.4) can be expanded directly in a Taylor series about $\epsilon = 0$ while the right side again can be expanded in a series of terms involving powers of ϵ^2 alone and powers of ϵ^2 multiplied by $\log\epsilon$. In this way (5.4) becomes (for $n \geqslant 1$)

$$\sum_{j=0}^{\infty}\frac{\epsilon^{2j}}{j\,!}\left[\left(\frac{d}{d\epsilon^2}\right)^j\left(2n\psi(\epsilon^2S(z),z)+4\epsilon^2S(z)\frac{\partial\psi}{\partial r^2}(\epsilon^2S(z),z)-\epsilon^2S'(z)\frac{\partial\psi}{\partial z}(\epsilon^2S(z),z)\right)\right]_{\epsilon=0}$$
$$\sim -\frac{1}{2\pi}\epsilon^{-2n}\frac{2^{n-1}}{n\binom{2n-1}{n}}\left(\frac{z(1-z)}{S(z)}\right)^n+\epsilon^{-2n+2}\sum_{j=0}^{\infty}\epsilon^{2j}\,(\tilde{L}_j+\log(\epsilon)\,\tilde{N}_j)f(z,\epsilon). \quad (5.5)$$

In (5.5), the $\tilde{L}_j$ and $\tilde{N}_j$ are linear operators which can be determined using the method of Handelsman & Keller and

$$\binom{2n-1}{n} = (2n-1)\,!/n\,!(n-1)\,!.$$

Equation (5.5) suggests that we look for a solution for $f(z,\epsilon)$ of the same form as (3.4), but with the factor ϵ^2 in front replaced by ϵ^{2n}. If this expansion is inserted into (5.5) we are led to a system of recursive equations similar to (3.5)–(3.7). In particular, we find that

$$f(z,\epsilon) \sim \frac{-\pi n^2}{2^{2n-3}}\binom{2n-1}{n}\left\{\frac{S(z)}{z(1-z)}\right\}^n\psi(0,z)\,\epsilon^{2n}+O(\epsilon^{2n+2}\log\epsilon). \quad (5.6)$$

A careful examination of the form of the operators $\tilde{L}_j$ and $\tilde{N}_j$ and the requirement that $\tilde{L}_jF(z)$ and $\tilde{N}_jF(z)$ be regular whenever $F(z)$ is regular leads to the same requirements on $\alpha(\epsilon)$ and $\beta(\epsilon)$ as in §3. This is done by integrating by parts several times the integrals which appear in $\tilde{L}_j$ and $\tilde{N}_j$ and then showing that they are in fact just certain linear combinations of integrals of the same type as those which appear in L_j and N_j in §3. Hence, $\alpha(\epsilon)$ and $\beta(\epsilon)$ are again exactly as determined in Handelsman & Keller and the leading terms in their expansions are again given by (3.8) and (3.9). (See appendix B.)

6. Discussion of the method

The method of Handelsman & Keller (1967*a*) has now been used by several authors (see references above and also Handelsman & Keller 1967*b*) to obtain the uniform asymptotic expansion of the solution of a linear integral equation in which the kernel becomes singular as ϵ approaches zero. We can now make some general observations about this method.

First, for three-dimensional problems involving a slender body of revolution, the density of the distribution of singularities (i.e. the primary unknown in our integral equation) can be assumed to be a regular function of z for $0 \leqslant z \leqslant 1$, while it is not regular in ϵ near $\epsilon = 0$, since terms involving $\log \epsilon$ appear. From (5.4), we see that, as ϵ approaches zero, the kernel behaves like the sum of terms which are $O((z-\xi)^{-2n-1})$ and $O(\epsilon^2(z-\xi)^{-2n-3})$. For $n = 0$ this singularity is not 'too severe' and the method works fine. However, for $n \geqslant 1$, this singularity apparently becomes too strong for the method to work and hence an adjustment in the form of the solution [e.g. (3.1)] must be made. More precisely, we must assume that the solution vanishes at each end point of its distribution, like the distance from the end point to the nth power [see (5.4)]. If this assumption is not made, the method will simply not yield a uniform asymptotic solution of the integral equation (see appendix B).

Second, for two-dimensional problems (Geer 1974; Geer & Keller 1968), the denominator in the kernel looks just like $(z-\xi)^2+\epsilon^2 S(z)$ and, in particular, not this quantity raised to a fractional power. As ϵ approaches zero, the kernel involves terms which are $O(\xi - z)^{-1}$ and $O(\epsilon(z-\xi)^{-2})$. In this case, only integer powers of ϵ appear in the expansion of the solution, but the solution is not a regular function of z. In particular, near each end point of its distribution, the solution blows up like the inverse square root of the distance from the end point. Again, unless the form of the solution is modified to take into account this singularity, the method will not yield a uniform expansion of this solution.

Finally, it is interesting just to note that the extent of the distribution (i.e. the $\alpha(\epsilon)$ and $\beta(\epsilon)$) is exactly the same for all the two- and three-dimensional problems mentioned above which involve a symmetric body. The only exception here is the asymmetric two-dimensional problem treated in Geer (1974), where α and β are modified slightly. The full significance of this observation escapes this author at present, but at least we should not be surprised when it happens again.

The author wishes to express thanks to Joseph B. Keller of N.Y.U. for suggesting this problem and for offering helpful suggestions concerning its solution.

Appendix A. The operators L_j and N_j

In this appendix we present the formulae for the linear operators L_j and N_j which appear in the expansion (3.3). More precisely, they are the coefficients in the expansion of the operator

$$I(z,\epsilon) \equiv \int_{\alpha(\epsilon)}^{\beta(\epsilon)} \left\{ \frac{1}{[(z-\xi)^2+\epsilon^2 S(z)]^{\frac{3}{2}}} - \tfrac{3}{2}\epsilon^2 \frac{2S(z)-S'(z)(z-\xi)}{[(z-\xi)^2+\epsilon^2 S(z)]^{\frac{5}{2}}} \right\} (\beta(\epsilon)-\xi)(\xi-\alpha(\epsilon))F(\xi)\,d\xi$$

$$\sim \frac{2z(1-z)}{S(z)}\epsilon^{-2}F(z) + \sum_{j=0}^{\infty} \epsilon^{2j}(L_j+\log(\epsilon)\,N_j)\,F(z), \qquad \text{(A 1)}$$

where $F(z)$ is assumed to be a regular function of z on $0 \leqslant z \leqslant 1$ and is independent of ϵ.

The expansion of $I(z,\epsilon)$ follows closely along the lines given in Handelsman & Keller. In particular, it is again necessary to break up the interval of integration into an interval from $\alpha(\epsilon)$ to z and one from z to $\beta(\epsilon)$. Several of the resulting integrals can be simplified by integration by parts and noting that the integrand vanishes at $\alpha(\epsilon)$ and $\beta(\epsilon)$. Assuming that $\alpha(\epsilon)$ and $\beta(\epsilon)$ have expansions of the form

$$\alpha(\epsilon) = \sum_{j=1}^{\infty} \alpha_j \epsilon^{2j}, \quad \beta(\epsilon) = 1 - \sum_{j=1}^{\infty} \beta_j \epsilon^{2j}, \tag{A 2}$$

where $\alpha_0 = 0$ and $\beta_0 = 1$, the final form of the operators is the following:

$$\begin{aligned} L_j(F(z)) \equiv & \sum_{k=0}^{j-1} \hat{F}^{2k+2}(z) \frac{d}{dz} (G_{j-k}^{k+1,1}(z) - R_{j-k}^{k+1,1}) + \sum_{k=0}^{j} \Big\{ \hat{F}^{2k+2}(z)\,(2k+1)\,(G_{j-k}^{k+1,0} - R_{j-k}^{k+1,0}) \\ & + \hat{F}^{2k+1}(z)\,[d(G_{j-k}^{k+1,0} - R_{j-k}^{k+1,0})/dz + 2k(G_{j-k}^{k,1} - R_{j-k}^{k,1})] \\ & + \alpha_k d\{[S(z)]^k h_{1,j-k}^k\}/dz - \alpha_k (S(z))^k h_{0,j-k}^k \Big\} + M_j(z)\,F(z) \end{aligned} \tag{A 3}$$

and

$$\begin{aligned} N_j(F(z)) = & \sum_{k=0}^{j-1} \hat{F}^{2k+2}(z) \frac{d}{dz} D_{j-k}^{k+1,1}(z) \\ & + \sum_{k=0}^{j} \left\{ \hat{F}^{2k+2}(z)\,(2k+1)\,D_{j-k}^{k+1,0} + \hat{F}^{2k+1}(z) \left[\frac{d}{dz} D_{j-k}^{k+1,0} + D_{j-k}^{k,1} \right] \right\} + \delta_{j,0} 2F(z). \end{aligned} \tag{A 4}$$

In (A 3) and (A 4), $\hat{F}^m(z) \equiv F^{(m)}(z)/m!$ and

$$\begin{aligned} h_{k,j}^p(z,F) = & \frac{1}{j!} \left[\left(\frac{d}{d\epsilon^2} \right)^j \left\{ \left(\int_0^{\beta(\epsilon)-z} - \int_{\alpha(\epsilon)-z}^0 \right) v^{-2p-3+k} \left(F(z+v) - \sum_{s=0}^{2p+2-k} \hat{F}^s(z)\, v^s \right) \right.\right. \\ & \left.\left. \times (\beta(\epsilon) - v - z)\,(v + z - \alpha(\epsilon))\,dv \right\} \right]_{\epsilon=0}, \end{aligned} \tag{A 5}$$

$$\begin{aligned} R_j^{p,l}(z) = & \sum_{k=0}^{\min(j,p-2+l)} \alpha_k [S(z)]^k \left\{ \frac{\tilde{T}_{2p+l-2k,j-k} + T_{2p+l-2k,l-k}}{(2p+l-2k)\,(2p+l-2k-1)} \right. \\ & \left. - \sum_{s=0}^{j-k} \frac{T_{1,s} \tilde{T}_{2p+l-2k-1,j-k-s} + \tilde{T}_{1,s} T_{2p+l-2k-1,j-k-s}}{(2p+l-2k-1)\,(2p+l-2k-2)} \right\}, \end{aligned} \tag{A 6}$$

$$\begin{aligned} G_j^{p,l}(z) = & -(2p+l+1)\,K_j^{2p+l}(z) \\ & + (2p+l) \sum_{k=0}^{j} K_{j-k}^{2p+l-1} (\tilde{T}_{1,k} + T_{1,j}) - (2p+l-1) \sum_{s=0}^{j} K_{j-s}^{2p+l-2} \sum_{k=0}^{s} \tilde{T}_{1,k} T_{1,s-k}, \end{aligned} \tag{A 7}$$

$$\begin{aligned} M_j(z) = & 2g_j'(z) - 2h_j'(z) - J_j(z) - 2zJ_j'(z) + \sum_{k=0}^{j} (\beta_k + \alpha_k)\,J_{j-k}'(z) \\ & - \frac{1}{2S(z)} \sum_{k=0}^{j+1} (g_k(z) - g_k(0) - h_k(z) + h_k(0)) \\ & \times (2\delta_{j+1-k,0} + g_{j+1-k}(z) - g_{j+1-k}(0) + h_{j+1-k}(z) - h_{j+1-k}(1)), \end{aligned} \tag{A 8}$$

where

$$K_j^n(z) = \sum_{k=0}^{\min(j,[\frac{1}{2}(n-1)])} b_{n,k} \frac{[-S(z)]^k}{n-2k} \sum_{s=0}^{j-k} (h_{j-k-s} \tilde{T}_{n-1-2k,s} - g_{j-k-s} T_{n-1-2k,s}), \tag{A 9}$$

$$T_{k,j}(z) = \frac{1}{j!} \left[\left(\frac{d}{d\epsilon^2} \right)^j (\alpha(\epsilon) - z)^k \right]_{\epsilon=0}, \quad \tilde{T}_{k,j}(z) = \frac{1}{j!} \left[\left(\frac{d}{d\epsilon^2} \right)^j (\beta(\epsilon) - z)^k \right]_{\epsilon=0}, \tag{A 10}$$

$$J_k(z) = \begin{cases} \log\{4z(1-z)/S(z)\}, & k=0, \\ \dfrac{1}{k!}\left[\left(\dfrac{d}{d\epsilon^2}\right)^k \log\left\{\dfrac{1+\sum\limits_{j=1}^{\infty}\epsilon^{2j}\left[\dfrac{h_j(z)-h_j(1)}{2(1-z)}\right]}{1+\dfrac{2z}{S(z)}\sum\limits_{j=1}^{\infty}\epsilon^{2j}[g_{j+1}(z)-g_{j+1}(1)]}\right\}\right]_{\epsilon=0}, & k \geqslant 1, \end{cases} \tag{A 11}$$

$$g_j(z) = \frac{1}{j!}\left\{\left(\frac{d}{d\epsilon^2}\right)^j \{(\alpha(\epsilon)-z)^2+\epsilon^2 S(z)\}^{\frac{1}{2}}\right\}_{\epsilon=0}, \tag{A 12}$$

$$h_j(z) = \frac{1}{j!}\left[\left(\frac{d}{d\epsilon^2}\right)^j \{(\beta(\epsilon)-z)^2+\epsilon^2 S(z)\}^{\frac{1}{2}}\right]_{\epsilon=0}. \tag{A 13}$$

The constants a_j and $b_{n,j}$ are defined by

$$a_0 = 1, \quad a_j = \frac{(-1)^j \times 3 \times 5 \times \ldots \times (2j+1)}{j! \times 2^j}, \quad j \geqslant 1, \tag{A 14}$$

$$b_{n,j} = \begin{cases} 1, & j=0, \\ \prod\limits_{i=1}^{j-1}\dfrac{n-1-2i}{n-2i}, & j = 1,2,3,\ldots, \\ 0, & j \text{ not an integer.} \end{cases} \tag{A 15}$$

Here $[n]$ denotes the greatest integer not exceeding n. The $D_j^{p,l}(z)$ are defined by the right side of (A 7), except that K_j^n is replaced by $\tilde{K}_j^n$, where

$$\tilde{K}_j^n = \begin{cases} 0, & j < \frac{1}{2}n, \\ -2b_{n,\frac{1}{2}n}[(-S(z))^{\frac{1}{2}n} J_{j-\frac{1}{2}n}(z), & j \geqslant \frac{1}{2}n. \end{cases} \tag{A 16}$$

We note that all of these expressions will be regular on $0 \leqslant z \leqslant 1$ whenever $F(z)$ is, provided that the functions $g_j(z)$ and $h_j(z)$ are regular on $0 \leqslant z \leqslant 1$. This leads to the same requirements on $\alpha(\epsilon)$ and $\beta(\epsilon)$ as in Handelsman & Keller (1967*a*). The operators $\tilde{L}_j$ and $\tilde{N}_j$ which appear in (5.5) have the same general form as the operators L_j and N_j above, although the individual expressions are longer and more involved.

Appendix B

In this appendix we outline a proof that the integral operator on the right side of (5.4) has a regular expansion if $\alpha(\epsilon)$ and $\beta(\epsilon)$ are chosen in exactly the same manner as in Handelsman & Keller (1967*a*). More precisely, since the integral operator in (5.4), operating on a regular function $F(z)$, independent of ϵ, can be written as

$$\frac{1}{2\pi}\left\{\frac{d}{dz} W_1^n(z,\epsilon) - n W_0^n(z,\epsilon)\right\}, \tag{B 1}$$

where $$W_j^n(z,\epsilon) = \int_{\alpha(\epsilon)}^{\beta(\epsilon)} \frac{(\xi-z)^j}{[\epsilon^2 S(z)+(z-\xi)^2]^{n+\frac{1}{2}}}(\xi-\alpha(\epsilon))^n(\beta(\epsilon)-\xi)^n F(\xi)\,d\xi, \tag{B 2}$$

it is sufficient to prove that each $W_j^n(z,\epsilon)$, $j = 0,1$, $n = 0,1,\ldots$, has a regular expansion if the $\alpha(\epsilon)$ and $\beta(\epsilon)$ are chosen as in Handelsman & Keller.

Our proof will be by induction. From the work presented in Handelsman & Keller and in appendix A above, it follows that $W_j^0(z,\epsilon)$ and $W_j^1(z,\epsilon)$, $j = 0, 1$, all have regular expansions. Now suppose that $W_j^0, W_j^1, \ldots, W_j^k$, $j = 0, 1$, all have regular expansions with $\alpha(\epsilon)$ and $\beta(\epsilon)$ chosen as in Handelsman & Keller. We now show that W_0^{k+1} and W_1^{k+1} have regular expansions by showing that they can be expressed as certain linear combinations of the W_j^n, $j = 0, 1$, $n = 0, 1, \ldots, k$. Using the fact that

$$\frac{\xi - z}{[\epsilon^2 S(z) + (z-\xi)^2]^{k+\frac{3}{2}}} = -\frac{1}{2k+1}\frac{d}{d\xi}\left\{\frac{1}{[\epsilon^2 S(z) + (z-\xi)^2]^{k+\frac{1}{2}}}\right\}$$

we integrate by parts the expression for W_1^{k+1} from (B 2) and obtain

$$W_1^{k+1}(z,\epsilon) = \frac{1}{2k+1}\int_{\alpha(\epsilon)}^{\beta(\epsilon)} \frac{1}{[\epsilon^2 S(z) + (\xi - z)^2]^{k+\frac{1}{2}}}(\xi - \alpha(\epsilon))^k (\beta(\epsilon) - \xi)^k$$
$$\times \{(k+1)F(\xi)(\alpha(\epsilon) + \beta(\epsilon) - 2\xi) + (\beta(\epsilon) - \xi)(\xi - \alpha(\epsilon))F'(\xi)\}\,d\xi. \quad \text{(B 3)}$$

In obtaining (B 3) we have used the fact that the integrand vanishes at $\xi = \alpha(\epsilon)$ and $\xi = \beta(\epsilon)$. By our induction hypothesis on W_0^k, the right side of (B 3) has a regular expansion and hence so does $W_1^{k+1}(z,\epsilon)$.

To show that $W_0^{k+1}(z,\epsilon)$ has a regular expansion, we note that

$$\frac{1}{[\epsilon^2 S(z) + (\xi - z)^2]^{k+\frac{3}{2}}} = \frac{d}{d\xi}\left\{[\epsilon^2 S(z)]^{-k-1}\sum_{j=0}^{k} C_j \frac{(\xi - z)^j}{[\epsilon^2 S(z) + (z-\xi)^2]^{j+\frac{1}{2}}}\right\}, \quad \text{(B 4)}$$

where the C_j are certain constants. We now integrate by parts the expression (B 2) with $j = 0$ and $n = k+1$ and obtain

$$W_0^{k+1}(z,\epsilon) = -[\epsilon^2 S(z)]^{-k-1}\sum_{j=0}^{k} C_j \int_{\alpha(\epsilon)}^{\beta(\epsilon)} \frac{(\xi - z)^j}{[\epsilon^2 S(z) + (z-\xi)^2]^{j+\frac{1}{2}}}$$
$$\times (\xi - \alpha(\epsilon))^k (\beta(\epsilon) - \xi)^k \{(k+1)F(\xi)(\alpha(\epsilon) + \beta(\epsilon) - 2\xi)$$
$$+ (\beta(\epsilon) - \xi)(\xi - \alpha(\epsilon))F'(\xi)\}\,d\xi. \quad \text{(B 5)}$$

Equation (B 5) expresses W_0^{k+1} as a linear combination of integrals of the form of $W_j^j(z,\epsilon)$ for $j = 0, 1, \ldots, k$. But now using the fact that we can write

$$\frac{(\xi - z)^j}{[\epsilon^2 S(z) + (z-\xi)^2]^{j+\frac{1}{2}}} = \frac{(\xi - z)^{j-2}}{[\epsilon^2 S(z) + (z-\xi)^2]^{j-\frac{1}{2}}} - \epsilon^2 S(z)\frac{(\xi - z)^{j-2}}{[\epsilon^2 S(z) + (z-\xi)^2]^{j+\frac{1}{2}}}$$

and then repeating this process as often as necessary, we can express each $W_j^j(z,\epsilon)$ as a linear combination of integrals of the form W_0^s and W_1^s, for $s = 0, 1, \ldots, j$. Thus, ultimately, since the summation in (B 5) extends only up to $j = k$, the right side of (B 5) can be expressed as a linear combination of integrals of the form of W_0^n and W_1^n, $n = 0, 1, \ldots, k$. Hence, by our induction hypothesis, W_0^{k+1} has a regular expansion. This completes our proof.

We should now make one final observation that if the integrand does not vanish in (5.4), i.e. if the factor $(\beta(\epsilon) - \xi)^n(\xi - \alpha(\epsilon))^n$ is omitted, we cannot make the resulting expansions regular by any choice of the constants α_n and β_n in (A 2). To illustrate this point, if we look at $W_1^1(z,\epsilon)$ with the factor

$$(\beta(\epsilon) - \xi)(\xi - \alpha(\epsilon))$$

removed we can integrate this expression by parts and express W_1^1 in terms of W_0^0 and the terms

$$\frac{F(\alpha(\epsilon))}{[\epsilon^2 S(z)+(z-\alpha(\epsilon))^2]^{\frac{1}{2}}}, \quad \frac{F(\beta(\epsilon))}{[\epsilon^2 S(z)+(z-\beta(\epsilon))^2]^{\frac{1}{2}}}. \tag{B 6}$$

Using the expansion (A 2) it follows that the expansions of the terms in (B 6) will be singular at $z = 0$ or $z = 1$, unless they vanish identically. That is, no choice of the constants α_n and β_n can prevent these singularities if

$$F(\beta(\epsilon)) \neq 0 \neq F(\alpha(\epsilon)).$$

Hence, it appears that our particular choice of $\alpha(\epsilon)$ and $\beta(\epsilon)$ and the requirement that our density functions vanish at $\alpha(\epsilon)$ and $\beta(\epsilon)$ are both necessary and sufficient conditions for our integrals to have regular expansions.

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